Math 3270 B

Tutorial 5.21

* Exact equation
* Exact equation with integrating factor

① $y(1+x^2)\,dx + y^2\,dy = 0 \quad \cdots (\clubsuit)$

Sol: First we see that $M = y(1+x^2)$, $M_y = 1+x^2$

$N = y^2$, $N_x = 0$, $M_y \neq N_x$, not exact.

Rewrite $(\clubsuit)$ as: $y\left((1+x^2)\,dx + y\,dy\right) = 0$

Hence $y \equiv 0$ or $(1+x^2)\,dx + y\,dy = 0$

$y \equiv 0$ is the trivial solution.

$\tilde{M} = 1+x^2$, $\tilde{N} = y$, $\tilde{M}_y = \tilde{N}_x = 0$, exact,

Then $\Phi_x = \tilde{M}$, $\Phi = x + \frac{1}{3}x^3 + g(y)$.

$\Phi_y = g'(y) = \tilde{N} = y \Rightarrow g(y) = \frac{1}{2}y^2$

Hence $\Phi(x,y) = x + \frac{1}{3}x^3 + \frac{1}{2}y^2 = C$, C is a constant.

is another solution to $(\clubsuit)$

$\square$.

② $2x^2y^3 + x(1+y^2)y' = 0, \quad \mu(x,y) = \frac{1}{xy^3}$

$M\mu = 2x, \quad N\mu = \frac{1+y^2}{y^3}$

$(M\mu)_y = 0 = (N\mu)_x,$

hence $\underbrace{2x}_{\tilde{M}} + \underbrace{\frac{1+y^2}{y^3}}_{\tilde{N}} y' = 0$ is exact

$\Phi_x = \tilde{M} = 2x$, then $\Phi = x^2 + g(y)$.

$\Phi_y = g'(y) = \frac{1}{y^3} + \frac{1}{y} \Rightarrow g(y) = -\frac{1}{2}y^{-2} + \ln|y|$

Hence $\Phi(x,y) = x^2 - \frac{1}{2}y^{-2} + \ln|y|$ is a solution.

Also, $y \equiv 0$ is a trivial solution.

③ $1+\left(\frac{x}{y}-\cos y\right)y'=0$

Sol: $M=1$, $M_y=0$, $N=\frac{x}{y}-\cos y$, $N_x=\frac{1}{y}$.

Hence we have $\frac{N_x-M_y}{M}=\frac{1}{y}$. only depends on y,

then $\mu'(y)=\frac{1}{y}\mu(y)$,

take $\mu(y)=y$.

And $\underset{\tilde{M}}{y}+\underset{\tilde{N}}{(x-y\cos y)}\,y'=0$

We see that $\tilde{M}_y=1=\tilde{N}_x$. This is an exact equation.

$\Phi_x=\tilde{M}=y \Rightarrow \Phi=xy+g(y)$.

$\Phi_y=x+g'(y)=x-y\cos y$.

Then $g(y)=-y\sin y-\cos y$.

Hence $\Phi(x,y)=xy-y\sin y-\cos y=C$,

where C is a constant

is the solution to the ODE.

□.

④. $3x+\frac{6}{y}+\left(\frac{x^2}{y}+\frac{3y}{x}\right)y'=0$ hint: $\mu(x,y)=\mu(xy)$

Sol: $M=3x+\frac{6}{y}$, $N=\frac{x^2}{y}+\frac{3y}{x}$

$M_y=-\frac{6}{y^2}$, $N_x=\frac{2x}{y}-\frac{6y}{x^2}$ (1)

We see that $M_y\neq N_x$, and $\frac{M_y-N_x}{N}$, $\frac{N_x-M_y}{M}$ both depend on x, y.

Now we consider $\mu(x,y)=\mu(xy)$

$$\partial_x\mu=y\mu',\quad \partial_y\mu=x\mu' \quad \cdots(2)$$

If $\mu(xy)$ is an integrating factor to the equation, then we must have:

$$(M\mu)_y=(N\mu)_x$$

i.e. $$M_y\mu+M\mu_y=N_x\mu+N\mu_x \quad \cdots(3)$$

Use (2) we have

$$M_y\mu+Mx\mu'=N_x\mu+Ny\mu'$$

Use (1) and after direct computation (3) becomes:

$$\mu'(xy)=\frac{1}{xy}\mu(xy),\quad \text{denote } z:=xy,$$

we see that $\mu'(z)=\frac{1}{z}\mu(z)$.

Hence take $\mu(z)=z$, i.e. $\mu(xy)=xy$.

Then $\underbrace{3x^2y + 6x}_{\tilde{M}} + (\underbrace{x^3 + 3y^2}_{\tilde{N}})\, y' = 0$

Then $\tilde{M}_y = 3x^2 = \tilde{N}_x$, this equation becomes exact.

$$\Phi_x = \tilde{M} \Rightarrow \Phi = x^3y + 3x^2 + g(y)$$

$$\Phi_y = x^3 + g'(y) = \tilde{N} \Rightarrow g'(y) = 3y^2,$$

$$g(y) = y^3$$

Hence $\Phi(x, y) = x^3y + 3x^2 + y^3 = C$, C is constant

is a solution to the ODE.

$\square$.